

# General non-Markovian dynamics of open quantum systems

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We present a general theory of non-Markovian dynamics for open quantum systems. We explore the non-Markovian dynamics by connecting the exact master equations with the non-equilibrium Green functions. Environmental back-actions are fully taken into account. The non-Markovian dynamics consists of non-exponential decays and dissipationless oscillations. Non-exponential decays are induced by the discontinuity in the imaginary part of the self-energy corrections. Dissipationless oscillations arise from band gaps or the finite band structure of spectral densities. The exact analytic solutions for various non-Markovian environments show that the non-Markovian dynamics can be largely understood from the environmental-modified spectra of the open systems.

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Understanding the dynamics of open quantum systems is one of the most challenging topics in physics, chemistry, and biology. The environment-induced quantum dissipation and decoherence dynamics are the main concerns in the study of open quantum systems [1, 2]. Decoherence control has also recently become a key task for practical implementations of nanoscale solid-state quantum information processing [3], where the decoherence is mainly dominated by non-Markovian dynamics due to the strong back-actions from the environment. A fundamental issue is how to accurately take into account non-Markovian memory effects, which have attracted considerable attention very recently both in theory [4–12] and in experiments [13–15].

The non-Markovian dynamics of an open quantum system can be described by the master equation of the reduced density matrix  $\rho(t)$ . This is obtained by tracing over the environmental degrees of freedom,  $\rho(t) = \text{tr}[\rho_{\text{tot}}(t)]$ , where  $\rho_{\text{tot}}(t)$  is the density matrix of the total system. The standard approach to the non-Markovian dynamics uses the Nakajima-Zwanzig operator projective technique [16] where the master equation is formally written as

$$\frac{d\rho(t)}{dt} = \int_{t_0}^t d\tau \mathcal{K}(t - \tau) \rho(\tau). \quad (1)$$

The non-Markovian memory effects are taken into account by the time non-local integral kernel  $\mathcal{K}(t - \tau)$ . In practice, very few systems can be exactly solved from (1). Therefore, the generality of non-Markovian dynamics has not been fully understood.

In general, there are three typical timescales in an open system to characterize non-Markovian dynamics: (i) the timescale of the system  $\sim 1/\varepsilon_s$ , where  $\varepsilon_s$  is a typical energy scale of that system; (ii) the timescale of the environment  $\sim 1/d$ , where  $d$  is the bandwidth of the en-

vironmental spectral density; (iii) the mutual timescale arising from the coupling between the system and the environment  $\sim 1/\Gamma$ , where  $\Gamma$  is the dominant coupling strength. It is usually believed that non-Markovian memory effects strongly rely on the relations among these different timescales. However, such relationships have not been quantitatively established yet.

Here, we show that the general behavior of non-Markovian dynamics is mainly determined by the band structure of the environment and the coupling between the system and the environment. We explore the non-Markovian dynamics from the analytical solution, solved by connecting the exact master equation with the non-equilibrium Green functions. Exact master equations have been derived only for limited system-environment couplings [7, 17–19]. Establishing the connection between the master equation and the non-equilibrium Green functions provides a new way to explore the non-Markovian dynamics even if the exact master equation of the open system is unknown.

*Exact master equation and non-equilibrium Green functions.*—We begin with a fermionic (bosonic) many-body system consisting of  $N$  single-particle energy levels  $\varepsilon_i$  ( $i = 1, 2, \dots, N$ ), coupled, via particle-particle exchanges, to a fermionic (bosonic) environment,  $H_{SB} = \sum_{\alpha ki} [V_{\alpha ki} a_i^\dagger b_{\alpha k} + V_{\alpha ki}^* b_{\alpha k}^\dagger a_i]$ . The environment can contain many different reservoirs, each reservoir is specified by its spectral density  $J_{\alpha ij}(\omega) = 2\pi \sum_k V_{\alpha ki} V_{\alpha kj}^* \delta(\omega - \varepsilon_k)$ , where  $V_{\alpha ki}$  is a coupling strength between the system and reservoir  $\alpha$ . The operators  $a_i^\dagger$  ( $a_i$ ) and  $b_{\alpha k}^\dagger$  ( $b_{\alpha k}$ ) are the particle creation (annihilation) operators of the discrete energy level  $i$  of the system and the continuous level  $k$  of reservoir  $\alpha$ , respectively. These creation-annihilation operators obey the standard anticommutation (commutation) relationship for fermions (bosons). Nonlinear particle-particle interactions in the system are not included. Using the coherent-state path-integral method [20] to the Feynman-Vernon influence functional [21], the exact master equation of such an open system can be de-

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rived [7, 18, 19]:

$$\begin{aligned} \frac{d\rho(t)}{dt} = & \frac{1}{i} \left[ \tilde{H}_S(t), \rho(t) \right] + \sum_{ij} \left\{ \gamma_{ij}(t) \left[ 2a_j \rho(t) a_i^\dagger \right. \right. \\ & - a_i^\dagger a_j \rho(t) - \rho(t) a_i^\dagger a_j \left. \right] + \tilde{\gamma}_{ij}(t) \left[ a_i^\dagger \rho(t) a_j \right. \\ & \left. \left. \pm a_j \rho(t) a_i^\dagger - a_i^\dagger a_j \rho(t) \mp \rho(t) a_j a_i^\dagger \right] \right\}. \end{aligned} \quad (2)$$

The first term in (2) is the unitary term with the renormalized Hamiltonian  $\tilde{H}_S(t) = \sum_{ij} \tilde{\epsilon}_{sij}(t) a_i^\dagger a_j$ . The second and third terms give the non-unitary dissipation and fluctuations, respectively. The  $\pm$  and  $\mp$  signs in the third term correspond to the system being bosonic/fermionic. The renormalized energy levels  $\tilde{\epsilon}_s(t)$ , the time-dependent dissipation coefficient  $\gamma(t)$  and the fluctuation coefficient  $\tilde{\gamma}(t)$  in (2) are given by

$$\tilde{\epsilon}_s(t) = \frac{i}{2} [\dot{\mathbf{u}}(t, t_0) \mathbf{u}^{-1}(t, t_0) - \text{H.c.}], \quad (3a)$$

$$\gamma(t) = -\frac{1}{2} [\dot{\mathbf{u}}(t, t_0) \mathbf{u}^{-1}(t, t_0) + \text{H.c.}], \quad (3b)$$

$$\tilde{\gamma}(t) = \dot{\mathbf{v}}(t, t) - [\dot{\mathbf{u}}(t, t_0) \mathbf{u}^{-1}(t, t_0) \mathbf{v}(t, t) + \text{H.c.}]. \quad (3c)$$

In Eqs. (3), the  $N \times N$  matrix functions  $\mathbf{u}(t, t_0)$  and  $\mathbf{v}(t, t)$  are related to the non-equilibrium Green functions of the system in the Schwinger-Keldysh non-equilibrium theory [22, 23],  $\mathbf{u}_{ij}(t, t_0) = \langle [a_i(t), a_j^\dagger(t_0)]_\mp \rangle$ , and  $\mathbf{v}_{ij}(t, t) = \langle a_j^\dagger(t) a_i(t) \rangle$  subtracting an initial-state dependent part [24]. These Green functions obey the Dyson equations,

$$\frac{d}{d\tau} \mathbf{u}(\tau, t_0) + i\epsilon_s \mathbf{u}(\tau, t_0) + \int_{t_0}^{\tau} d\tau' \mathbf{g}(\tau, \tau') \mathbf{u}(\tau', t_0) = 0, \quad (4a)$$

$$\begin{aligned} \frac{d}{d\tau} \mathbf{v}(\tau, t) + i\epsilon_s \mathbf{v}(\tau, t) + \int_{t_0}^{\tau} d\tau' \mathbf{g}(\tau, \tau') \mathbf{v}(\tau', t) \\ = \int_{t_0}^t d\tau' \tilde{\mathbf{g}}(\tau, \tau') \mathbf{u}^\dagger(\tau', t_0), \end{aligned} \quad (4b)$$

subjected to the boundary conditions  $\mathbf{u}(t_0, t_0) = 1$  and  $\mathbf{v}(t_0, t) = 0$  with  $t_0 \leq \tau \leq t$ , where  $\epsilon_s$  is a  $N \times N$  matrix given by the bare single-particle energy levels of the system. The self-energy corrections,  $\mathbf{g}(\tau, \tau')$  and  $\tilde{\mathbf{g}}(\tau, \tau')$ , which take into account all the back-actions from the environment, are expressed explicitly by

$$\mathbf{g}(\tau, \tau') = \sum_{\alpha} \int \frac{d\omega}{2\pi} \mathbf{J}_{\alpha}(\omega) e^{-i\omega(\tau-\tau')} , \quad (5a)$$

$$\tilde{\mathbf{g}}(\tau, \tau') = \sum_{\alpha} \int \frac{d\omega}{2\pi} \mathbf{J}_{\alpha}(\omega) f_{\alpha}(\omega) e^{-i\omega(\tau-\tau')} , \quad (5b)$$

where the function  $f_{\alpha}(\omega) = [e^{\beta_{\alpha}(\omega-\mu_{\alpha})} \mp 1]^{-1}$  is the Bose-Einstein (Fermi-Dirac) distribution of bosonic (fermionic) reservoir  $\alpha$  at the initial time  $t_0$ . Equations (2)-(5) establish a rigorous connection between the

known exact master equation and the non-equilibrium Green functions for open quantum systems.

*General non-Markovian dynamics.*—Different from the Nakajima-Zwanzig master equation, the exact master equation (2) is local in time, characterized by the dissipation and the fluctuation coefficients,  $\gamma(t)$  and  $\tilde{\gamma}(t)$ . Non-Markovian memory effects are manifested as follows:

(i) The coefficients  $\gamma(t)$  and  $\tilde{\gamma}(t)$  are microscopically and non-perturbatively determined by the non-equilibrium Green functions from the Dyson equations (4). The non-Markovian memory effect is fully coded into the homogenous non-local time integrals in (4) with the integral kernel  $\mathbf{g}(\tau, \tau')$ . In other words, the self-energy correction  $\mathbf{g}(\tau, \tau')$  serves as a memory kernel that count all the back-actions from the environment.

(ii) The coefficients  $\gamma(t)$  and  $\tilde{\gamma}(t)$  are constrained by the non-equilibrium fluctuation-dissipation theorem. The inhomogenous non-local time integral in (4b) with the integral kernel  $\tilde{\mathbf{g}}(\tau, \tau')$ , depicts the fluctuation arisen from the environment. Because  $\mathbf{v}(t_0, t) = 0$ , we can analytically solve Eq. (4b):

$$\mathbf{v}(\tau, t) = \int_{t_0}^{\tau} d\tau_1 \int_{t_0}^t d\tau_2 \mathbf{u}(\tau, \tau_1) \tilde{\mathbf{g}}(\tau_1, \tau_2) \mathbf{u}^\dagger(t, \tau_2). \quad (6)$$

This solution shows that Eq. (3c) is a generalized non-equilibrium fluctuation-dissipation theorem in the time domain (the reduction to the equilibrium fluctuation-dissipation theorem is given in [25]). The fluctuation-dissipation theorem is a consequence of the unitarity of the whole system. It guarantees the positivity of the reduced density matrix during the non-Markovian time evolution.

Based on the above intrinsic features of open quantum systems, we can now explore the general properties of non-Markovian dynamics. From Eqs. (3), we can express the Green function  $\mathbf{u}(t, t_0)$  in terms of the dissipation coefficient  $\gamma(t)$  as

$$\mathbf{u}(t, t_0) = \mathcal{T} \exp \left\{ - \int_{t_0}^t d\tau [i\tilde{\epsilon}(\tau) + \gamma(\tau)] \right\}, \quad (7)$$

where  $\mathcal{T}$  is the time-ordering operator. This solution indicates that  $\mathbf{u}(t, t_0)$  fully determines the dissipation dynamics of the system. However, due to the time-dependence of the dissipation coefficients, the detailed dissipation dynamics can vary significantly for different environments.

Explicitly, equation (5) show that  $\mathbf{g}(\tau, \tau') = \mathbf{g}(\tau - \tau')$  and  $\tilde{\mathbf{g}}(\tau, \tau') = \tilde{\mathbf{g}}(\tau - \tau')$ . Thus we can write  $\mathbf{u}(t, t_0) = \mathbf{u}(t - t_0)$ . Using the modified Laplace transform  $\mathbf{U}(z) = \int_{t_0}^{\infty} dt \mathbf{u}(t) e^{iz(t-t_0)}$ , it is easy to obtain

$$\mathbf{U}(z) = \frac{i}{z\mathbf{I} - \epsilon_s - \Sigma(z)}, \quad (8)$$

where  $\mathbf{I}$  is the identity,  $\Sigma(z)$  is the Laplace transform of

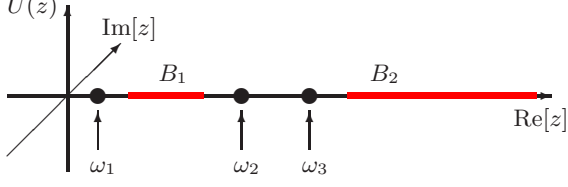


FIG. 1: (color online) A schematic pole structure of the Green function  $U(z)$ . The thick red lines on the real  $z$  axis correspond to  $\sum_{\alpha} J_{\alpha}(z) \neq 0$ .

the self-energy correction,

$$\Sigma(z) = \sum_{\alpha} \int \frac{d\omega}{2\pi} \frac{J_{\alpha}(\omega)}{z - \omega} \xrightarrow{z=\omega \pm i0^+} \Delta(\omega) \mp i \sum_{\alpha} \frac{J_{\alpha}(\omega)}{2}, \quad (9)$$

and  $\Delta(\omega) = \sum_{\alpha} \mathcal{P} \int \frac{d\omega'}{2\pi} \frac{J_{\alpha}(\omega')}{\omega - \omega'}$  is the principal value of the integral. It can be shown that the general solution of  $\mathbf{u}(t, t_0)$  is given by

$$\mathbf{u}(t - t_0) = \sum_i \mathcal{Z}_i e^{-i\omega_i(t-t_0)} + \sum_k \int_{B_k} \frac{d\omega}{2\pi} \left[ U(\omega + i0^+) - U(\omega - i0^+) \right] e^{-i\omega(t-t_0)}. \quad (10)$$

The first term in (10) corresponds to localized modes with poles  $\{\omega_i\}$  located at the real  $z$  axis with  $\sum_{\alpha} J_{\alpha}(\omega) = 0$ . The coefficients  $\{\mathcal{Z}_i\}$  are the corresponding residues. The localized modes exist only when the environmental spectral density has band gaps or a finite band, i.e.,  $\sum_{\alpha} J_{\alpha}(\omega)$  vanishes in some frequency regions, see Fig. 1. These localized modes do not decay, and give dissipationless non-Markovian dynamics. The second term in (10) is the contribution from the branch cuts  $\{B_k\}$ , due to the discontinuity of  $\Sigma(z)$ , so does  $U(z)$ , across the real axis on the complex space  $z$ , see Eq. (9). The branch cuts usually generate non-exponential decays [27], which is another significance of the non-Markovian dynamics. When the system is weakly coupled to the environment, the non-exponential decays are reduced to exponential-like decays.

Equation (10) provides indeed a general solution of the non-Markovian dissipation dynamics. It shows that the non-Markovian dissipation dynamics consists of non-exponential decays plus dissipationless localized modes. Such a solution for the two-point Green function  $\mathbf{u}(t, t_0)$  is generic and can be proven from the quantum field theory [33], even if particle-particle interactions are included.

The Green function  $\mathbf{u}(t, t_0)$  reveals the general non-Markovian dissipation dynamics. The non-Markovian fluctuation dynamics is constrained by the fluctuation-dissipation theorem via the Green function  $\mathbf{v}(t, t)$  of

(6). Thus, the whole picture of non-Markovian dynamics is fully characterized by the dissipation and fluctuation coefficients of (3). The non-exponential decay part of (10) makes the dissipation coefficient  $\gamma(t)$  oscillates between positive and negative values, representing the back-flow of information from the system to the environment [8, 9]. Non-exponential decays alone give  $\gamma(t)$  a non-zero asymptotical value. If there are localized modes,  $\gamma(t)$  will vanish in the steady-state limit, resulting in dissipationless oscillations. In the weak coupling region,  $\gamma(t)$  can still be time-dependent but keeps positive, the corresponding dynamics gives simple exponential-like decays, observed mainly in the Markovian limit. Furthermore Eqs. (3c) and (6) together show that except for the initial environmental temperature dependence, the time-dependence of the fluctuation coefficient  $\tilde{\gamma}(t)$  behaves similar to  $\gamma(t)$ , due to the fluctuation-dissipation theorem. In conclusion, non-Markovian dynamics can be fully understood from the solution of the Green function  $\mathbf{u}(t, t_0)$ .

*Examples and discussion* —To be more specific, let us first examine the non-Markovian dynamics of a single-mode bosonic nanosystem, such as a nanophotonic or optomechanical resonator, coupled to a general non-Markovian environment with spectral density

$$J(\omega) = 2\pi\eta\omega \left(\frac{\omega}{\omega_c}\right)^{s-1} \exp\left(-\frac{\omega}{\omega_c}\right), \quad (11)$$

where  $\eta$  is the coupling constant between the system and the environment, and  $\omega_c$  is the frequency cutoff. When  $s = 1, < 1$  and  $> 1$ , the corresponding environments are Ohmic, sub-Ohmic and super-Ohmic, respectively [26]. Following the above general procedure, the analytical solution of the non-Markovian dissipation dynamics is given by (setting  $t_0 = 0$  for simplicity):

$$u(t) = \mathcal{Z} e^{-i\omega' t} + \frac{2}{\pi} \int_0^{\infty} d\omega \frac{J(\omega) e^{-i\omega t}}{4[\omega - \varepsilon_s - \Delta(\omega)]^2 + J^2(\omega)}, \quad (12)$$

where  $\Delta(\omega) = \frac{1}{2}[\Sigma(\omega + i0^+) + \Sigma(\omega - i0^+)]$  and the Laplace transform of the self-energy correction

$$\Sigma(\omega) = \begin{cases} \eta\omega_c [\pi\sqrt{-\tilde{\omega}} e^{-\tilde{\omega}} \text{erfc}(\sqrt{-\tilde{\omega}}) - \sqrt{\pi}] & s = 1/2 \\ \eta\omega_c [\tilde{\omega} \exp(-\tilde{\omega}) \text{Ei}(\tilde{\omega}) - 1] & s = 1 \\ \eta\omega_c [\tilde{\omega}^3 e^{-\tilde{\omega}} \text{Ei}(\tilde{\omega}) - \tilde{\omega}^2 - \tilde{\omega} - 2] & s = 3 \end{cases} \quad (13)$$

with  $\tilde{\omega} = \omega/\omega_c$ . Due to the vanishing spectral density for  $\omega < 0$ , a localized mode at  $\omega' = \varepsilon_s - \Sigma(\omega') < 0$  occurs when  $\eta\omega_c \Gamma(s) > \varepsilon_s$ , here  $\Gamma(s)$  is a gamma function. The localized mode leads to the dissipationless process. The corresponding residue is  $\mathcal{Z} = [1 - \Sigma'(\omega')]^{-1}$ . This analytical solution precisely reproduces the exact numerical solution in the previous work [10]. Figure 2 shows that

for a small  $\eta$ , the dissipation dynamics is an exponential-like decay. The corresponding  $\gamma(t)$  and  $\tilde{\gamma}(t)$  are time-dependent but positive (corresponding to Markovian dynamics). When  $\eta \gtrsim 0.3$ , the non-exponential decay dominates, and  $\gamma(t)$  and  $\tilde{\gamma}(t)$  oscillate in positive and negative values with nonzero asymptotical values. When  $\eta \gtrsim 0.6$ , the localized state occurs, and  $u(t)$  does not decay to zero. Correspondingly,  $\gamma(t)$  and  $\tilde{\gamma}(t)$  asymptotically approach to zero.

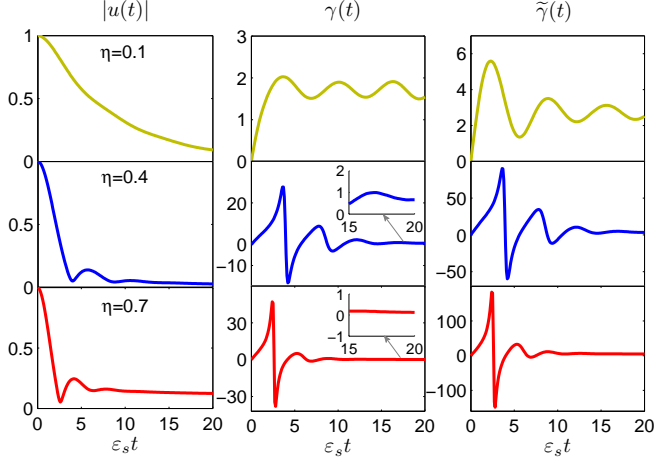


FIG. 2: (color online) The time evolution of the Green function  $u(t)$ , the dissipation and the fluctuation coefficients,  $\gamma(t)$  and  $\tilde{\gamma}(t)$ , in a sub-Ohmic bath, for several different values of the coupling constant  $\eta$ . We take other parameters  $\varepsilon_s = 13.83\mu\text{eV}$ ,  $\omega_c = \varepsilon_s$  and also  $k_B T = \varepsilon_s$ .

The second example is a fermionic system, a single electron transistor in nanostructures which consists of a quantum dot coupled to a source and a drain. The source and the drain are treated as two reservoirs of the environment. Their spectral densities take a Lorentzian form with a sharp cutoff,

$$J_\alpha(\omega) = \frac{\Gamma_\alpha d_\alpha^2}{(\omega - \omega_c)^2 + d_\alpha^2} \Theta(\Omega - |\omega - \omega_c|) \quad (14)$$

with  $\alpha = L(R)$  for the source (drain), where  $d_\alpha$  is the halfwidth of the spectral density and  $\Gamma_\alpha$  is the coupling strength between the system and reservoir  $\alpha$ . We add a sharp cutoff to simulate a finite band for the environmental density of states. When  $\Omega \rightarrow \infty$ , the above spectral density is reduced to the usual Lorentzian spectral density that has been used in various studies of nano-electronics [7, 28–31]. We consider the symmetric case,  $(\Gamma_L, d_L) = (\Gamma_R, d_R) = (\Gamma, d)$ . Then the analytical solution of the Green function  $u(t)$  becomes

$$u(t) = \sum_{j=1}^2 \mathcal{Z}_j e^{-i\omega'_j t} + \frac{1}{\pi} \int_{\omega_c - \Omega}^{\omega_c + \Omega} d\omega \frac{J(\omega) e^{-i\omega t}}{[\omega - \varepsilon_s - \Delta(\omega)]^2 + J^2(\omega)}, \quad (15)$$

where  $J(\omega) = J_L(\omega) = J_R(\omega)$  and  $\Delta(\omega)$  is the real part

of the self-energy  $\Sigma(\omega)$ ,

$$\Sigma(\omega) = \frac{J(\omega)}{\pi} \left[ \log \frac{\omega_c - \Omega - \omega}{\omega_c + \Omega - \omega} + \frac{2(\omega - \omega_c)}{d} \tan^{-1} \frac{\Omega}{d} \right]. \quad (16)$$

The two localized states are located outside of the band, i.e.,  $\omega'_j = \varepsilon_s + \Sigma(\omega'_j)$ , with  $\omega'_1 < \omega_c - \Omega$ , and  $\omega'_2 > \omega_c + \Omega$ . The corresponding residue is given by  $\mathcal{Z}_j = [1 - \Sigma'(\omega'_j)]^{-1}$ . Again, the localized modes lead to a dissipationless process and the integral term shows a non-exponential decay. Taking  $\Omega \rightarrow \infty$ , the two localized modes are excluded, and the solution of  $u(t)$  reproduces the exact non-Markovian dynamics of the usual Lorentzian spectral density (for detailed derivation, see [25]).

The third example is a two-level system involving single-photon processes (spontaneous emission). In general, a multi-level atomic open system does not obey the master equation (2). However, the Schrödinger equation of a two-level atomic system with only spontaneous single-photon emission processes (at zero temperature) can be reduced to the Dyson equation of (4a) [34–36]. For a two-level artificial atom, such as a quantum dot, embedded in photonic crystals, because of the photonic band gap it was shown [35] that the corresponding solution contains exponential decays, non-exponential decays, and localized bound modes all together. We find analytically [37] that the complex pole with exponential decay shown in [35] has been included in the branch-cut integral of (10). Explicitly, the spectral density of the photonic crystals takes  $J(\omega) = \frac{2C}{\sqrt{\omega - \omega_e}} \Theta(\omega - \omega_e)$  [34, 35]. From Eq. (10), we directly obtain the analytical solution of the spontaneous emission dynamics

$$u(t) = \frac{2\omega_r}{3\omega_r + \Delta} e^{i(\omega_r - \omega_e)t} + \frac{C}{\pi} \int_{\omega_e}^{\infty} d\omega \frac{\sqrt{\omega - \omega_e} e^{-i\omega t}}{(\omega - \varepsilon_s)^2 (\omega - \omega_e) + C^2}, \quad (17)$$

where  $\omega_r$  is the real root given by  $(\omega_r + \Delta)\sqrt{\omega_r} = C$ , and  $\Delta = \varepsilon_s - \omega_e$  is the detuning. This analytical solution recovers both the exact analytical and numerical solutions given in [34, 35].

The above examples show that very different open systems coupled to very different environments obey the same solution, Eq. (10), of the non-Markovian dynamics. The solutions of these examples can further be written in general as  $u(t - t_0) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathcal{D}(\omega) \exp\{-i\omega(t - t_0)\}$  with

$$\mathcal{D}(\omega) = 2\pi \sum_j \mathcal{Z}_j \delta(\omega - \omega'_j) + \frac{J(\omega)}{[\omega - \varepsilon_s - \Delta(\omega)]^2 + J^2(\omega)/4}. \quad (18)$$

Equation (18) shows that the environment modifies the system spectrum as a combination of localized modes (dissipationless process) plus a continuum spectrum part



(non-exponential decays). Remarkably, the result obtained from these simple examples gives indeed the general structure of two-point correlation functions in arbitrary complicated systems, see Ref. [33]. This indicates that alternatively, non-Markovian dynamics can be fully characterized by the environmental-modified spectrum of the system. If the spectrum of the open system can be measured, the non-Markovian dynamics can be extracted from its Fourier transform. This largely simplifies the exploration of the general properties of non-Markovian dynamics for more complicated open systems.

*Conclusion.*—By connecting the exact master equation with the non-equilibrium Green functions in many-body systems, we derive a general analytical solution of non-Markovian dynamics for open quantum systems, i.e., Eq. (10) or (18). From the analytical solution, we show that the general non-Markovian dynamics consists of non-exponential decays and dissipationless oscillations.

The dissipationless processes arise from band gaps or finite band structures of the environmental spectral densities. The non-exponential decays are induced by the discontinuity in the imaginary part of the self-energy corrections from the environment. The exponential decays observed in Markovian limit is a special case in the weak coupling limit. Since the non-equilibrium Green functions are well-defined for arbitrary quantum systems, this theory may also provide a new approach to explore non-Markovian dynamics for more complicated open systems whose exact master equation may be unknown.

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